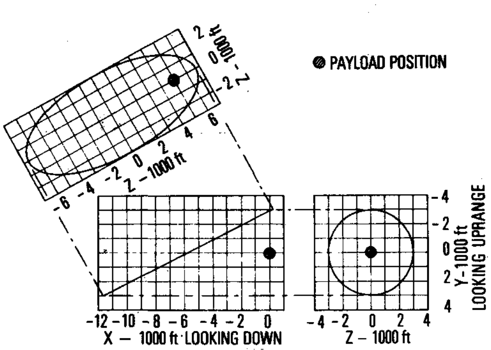

Fig. 4 Stationkeeping for worst phasing of vertical and horizontal motion and no initial phasing in downrange – relative orbit.

The other set of values for t_0, $\omega t_0 = \pi/2$, $3\pi/2$, etc., will now be considered. These values yield the maximum difference between the minimum and maximum separation. In this case,

$$r_{\text{min.}} = \dot{z}_0/\omega \qquad r_{\text{max.}} = \sqrt{4(\dot{z}_0/\omega)^2 + (\dot{y}_0/\omega)^2}$$

The difference between $r_{\text{min.}}$ and $r_{\text{max.}}$ can be minimized by eliminating the motion in the y direction completely ($\dot{y}_0 = 0$), in which case the maximum separation is twice the minimum value. This is just the motion in the vertical plane only.

Crossrange motion with an amplitude of 3000 ft will now be added. The phase of the crossrange motion relative to the vertical motion corresponds to that for maximum difference between minimum and maximum separation ($\omega t_0 = \pi/2$). This is illustrated in Fig. 3. Looking uprange, the relative orbit appears to be circular, although in true projection the orbit is elliptical. The increased range of separation distance is clearly evident.

Finally, if the initial phasing maneuver is eliminated from the stationkeeping initiation procedure, the motion along the flight path is entirely in the uprange direction, as shown in Fig. 4. This results in substantial increases in the maximum separation compared to the minimum. In addition, it results in the introduction of a component in the total separation distance with a period equal to the parking orbit period in addition to the half parking orbit period found previously.

Acknowledgment

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Choice of Method for Discretization of Continuous Systems

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Introduction

WHEN a digital computer is used to replace a continuous system for signal processing or control, it is necessary to convert the continuous process mathematical description into a discrete mathematical model, usually with difference equations. For example, digital filtering and simulation usually make this conversion necessary. The specific motivation for our work was digital autopilots, which may be classified as a real-time simulation application in that the goal was to replace existing hardware with a computer and software and then use a D/A (digital-to-analog) device to convert the numerical data back to a signal for aircraft control. The D/A converter, of course, is modeled as a zero-order hold.

A diagram which contains the essential functions is shown in Fig. 1. Input signals are sampled and converted to numbers; the linear transfer function to be implemented is described in a discrete manner with a method such as the z transform or Tustin or a numerical method such as Runge-Kutta. A data reconstruction device converts the computer output back into a continuous signal to operate the system. Only those situations described by Fig. 1 are of concern here, because the cases where output is used in the form of information have been the primary concern of the literature of discrete signal processing. And, judging from the literature, situations in which a signal output is required have received little attention. Perhaps this is because only recently has the revolution in small inexpensive computers placed an emphasis on systems working in real time whose outputs are converted to continuous signals.

In the case of an autopilot one could start over with new specifications and design a digital autopilot, but for existing systems it might be desired to simply discretize the existing transfer functions and make them function as nearly like the continuous system as needed for acceptance and satisfactory performance. This is the case of most digital filters, too, because of the rich body of classical filter design information that is available. Thus, the discretization problem where processed signals must be returned to the system is unavoidable; and it is easy to prove that, in general, it is not possible to find exact discrete equivalents to continuous systems. The principle differences of the output signals of a continuous processor and its digital replacement will be 1) phase shift and amplitude changes due to the hold device at

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the computer output; 2) phase shift and amplitude changes due to the digital representation of the continuous transfer function; 3) aliasing of high-frequency input components; and 4) unwanted components due to sampling.

Although these factors can result in major differences in the results of the two implementations, it should be possible to make digital-processed signals conform to commonly used specifications to within any accuracy desired if the sampling frequency of the digital system is made high enough; but, unfortunately, this parameter may be restricted to keep the sampling frequency low for computer capacity minimization and to minimize roundoff error. Thus, the sample frequency ω_s is of major importance.

Others have considered the discretization problem of continuous systems. Beale and Cook[1] used a bilinear transformation (Tustin operator[2]) and frequency warping in the manner of digital filter designers,[3] including all-pass structures for phase correction. They demonstrate good frequency response fidelity. Their work is primarily in the frequency domain, but they also obtained good time-domain performance. However, they do not take into account the effects of data reconstruction for those cases where computer output is converted to a signal for system use. Kuo et al.[4] have treated the problem in the time domain by forcing some states of the discretized system to be equal to corresponding states of the continuous system. They demonstrate the general inability to obtain exact equivalence between a continuous and discretized system but show that advantageous results can be achieved by matching some states. The problems associated with data reconstruction are not explicit in their paper. And, of course, a large body of information is available regarding simulation in which discretization is necessary.[2]

Comparison of Methods for Discretization

There are several methods available for the discretization process, including numerical methods, but work will be restricted to operational methods. Similar considerations are applicable to numerical methods and, of course, some operational and numerical methods are identical. The oldest and perhaps the most often used operational method is Tustin.[2,5] The Tustin operator requires that the Laplace transform variable s be replaced by a function of z such that

$$s=\frac{2}{T}\frac{z-1}{z+1} \tag{1}$$

Thus, for the transfer function

$$G(s)=1/(s+1) \tag{2}$$

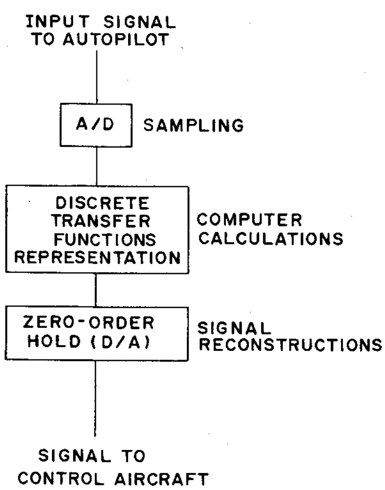

Fig. 1 Diagram of functions to realize digital autopilot.

Tustin's operator would yield

$$z_{\text{Tustin}}[G(s)]=G(z)=\frac{T(z+1)}{T(z+1)+2(z-1)} \tag{3}$$

Since the transfer function is the ratio of an output Y to an input U, Eq. (3) may be written as

$$\frac{Y(z)}{U(z)}=\frac{T(z+1)}{T(z+1)+2(z-1)} \tag{4}$$

or

$$[(T+2)+(T-2)z^{-1}]Y(z)=(T+Tz^{-1})U(z) \tag{5}$$

Using the translation theorem yields

$$y(nT)=\frac{-(T-2)}{T+2}y[(n-1)T]+\frac{T}{T+2}u(nT)+\frac{T}{T+2}u[(n-1)T] \tag{6}$$

This equation is programmed in the case of digital simulation of Eq. (2) or to realize a digital version of a filter described by Eq. (2). One might want to use frequency warping for more accurate results, but that is not the issue here. For other method definitions see Rosko[2] or Sage and Smith,[6] who also have a good presentation that includes a nonlinear example and some optimum results.

Figure 2 shows the discretization effect of some methods on the frequency response attenuation of the simple transfer function, Eq. (2). The Tustin, Boxer-Thaler, and Madwed methods are identical for a first-order operator, i.e. $1/s$. Note that the input frequency must be significantly lower than one-half the sampling error to have negligible discretization error. The zero-order hold attenuation is not as significant as attenuation due to discretization.

The phase angle at a given frequency is calculated for the discretized transfer function, then the phase shift of the continuous function (2) is subtracted, and their difference, θ, is plotted in Fig. 3. This phase difference is a price paid for discretization. The phase shift of a zero-order hold device is also shown for comparison purposes. Phase difference caused by discretization using the Tustin method is plotted in Fig. 4 for different values of A. The quantity A is defined as the product of the time constant and sampling frequency. Normally, the value of ω_s will be greater than twice the frequency of the pole involved, so $A>2$ would normally be the region of interest and at input frequencies less than $\omega<\omega_s/2$. This figure shows that the zero-order hold contributes much more phase shift than that due to discretization. More detailed information is available in Ref. 6; but from the information presented, it is apparent that the choice of discretization method is optional if a low enough sampling frequency is used to keep phase shift from a D/A converter relatively low.

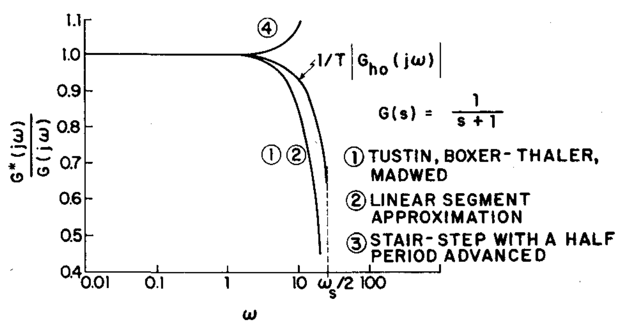

Fig. 2 Attenuation due to discretization for several methods (sampling frequency equals 16π rad/s) and due to zero-order hold.

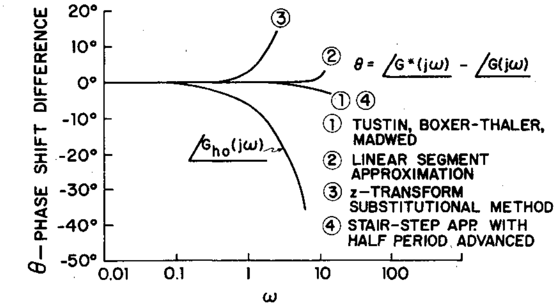

Fig. 3 Phase difference due to discretization for several methods (sampling frequency equals 16π) and phase shift for zero-order hold.

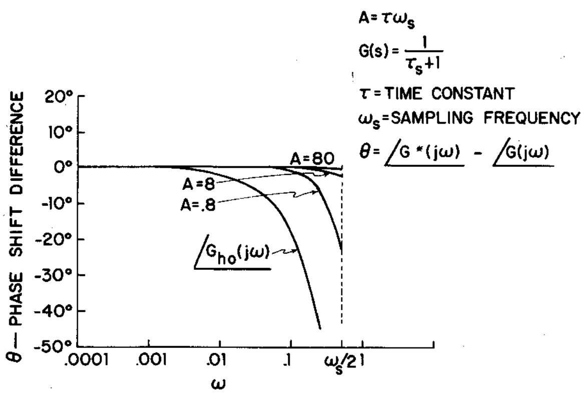

Fig. 4 Phase difference for difference sampling conditions and phase shift of zero-order hold.

Conclusions

From Fig. 3 it can be observed that a zero-order hold device introduces significantly more phase shift than all the methods considered for a simple transfer function. Similar results should prevail for complicated transfer functions. It is concluded that, in situations where a zero-order hold (i.e., D/A converter) must follow a computer, the method of discretization of a continuous system is not a major factor if phase shift is important. This is true because the sampling frequency must be relatively high to keep the phase shift of the hold device within acceptable limits, and this results in relatively small attenuation and phase shift due to discretization. For reasons of simplicity a logical choice is to use the Tustin method.

Acknowledgment

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Velocity Required for Intercept with Perturbations

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Introduction

AKNOWLEDGE of the velocity required to transfer a space vehicle in a J2 perturbing gravitational field from one position (ballistic missile or space vehicle) to another (target vector or target vehicle) is central in guidance and targeting theory. The space vehicle's thrust steering and cutoff equations generally depend on the velocity required and the instantaneous state of the vehicle obtained from the navigation system.

An analytic solution for v_{req} in a J2 field is presented[1]; however, the method could be applied in the case of arbitrary perturbations if these perturbations were known in terms of the variations of the classical elements. The theory was programmed and numerical results in terms of target miss are presented for several test cases.

The Closed-Form f and g Expressions

The fundamental expression employed in calculating the velocity required is the closed-form f and g vector equation for Keplerian motion. The equation development which follows is valid for all elliptic arcs.

$$r_T = fr + gv \tag{1}$$

where r_T is the target position in ECI (Earth-centered inertial coordinates), r is the initial position in ECI, v is the initial velocity in ECI, f, g are scaling coefficients.

$\delta\nu$ is the transfer true anomaly change:

$$f = 1 - \frac{r_T}{a(1-e^2)}(1-\cos\delta\nu) \tag{2a}$$

$$g = \frac{rr_T}{\sqrt{\mu a(1-e^2)}}\sin\delta\nu \tag{2b}$$

If no perturbations were present and the motion were Keplerian, the velocity required could be solved simply by inverting Eq. (1),

$$v_{\text{req}} \triangleq v = g^{-1}(r_T - fr) \tag{3}$$

However, J2 perturbations are assumed and therefore the focus is on specifying an auxiliary plane, here called the pseudo plane, and the pseudo instantaneous vehicle position which will allow use of the Keplerian equation (3). Equations are derived connecting the initial state osculating orbital quantities and the initial state pseudo orbital quantities.

Velocity Required in Terms of Pseudo Quantities

The plane of the final motion (r_T, v_T) is not, in general, the same as the osculating plane of the initial motion (r, v) and to use Keplerian analysis, a choice of plane must be made. In this development the pseudo plane is defined to be the plane of the final motion (r_T, v_T) and a pseudo initial position vector is specified, $\tilde{r}$, which lies in the pseudo plane. $\tilde{r}$ is

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